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Nonsymmetric Periodic Solutions of Certain Second Order Nonlinear Differential Equations*

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1. INTRODUCTION

Let $g(x)$ be a sufficiently regular odd function and let $f(t)$ be an even, 2π-periodic function which is also *odd-harmonic*, i.e. such that $f(t + \pi) \equiv -f(t)$. We shall say for brevity and to avoid the awkward phrase "even odd-harmonic", that $f(t)$ belongs to the class $\mathscr{E}$. [The term odd-harmonic is due to the fact that the Fourier series of such functions contain only odd harmonics.] The differential equation

$$x'' + g(x) = f(t) \tag{1.1}$$

may be expected to have periodic solutions in $\mathscr{E}$, which we shall call $\mathscr{E}$-solutions. Indeed, when $g(x)$ is linear and not of the form n^2x with n an integer, (1.1) has a unique 2π-periodic solution, which is in $\mathscr{E}$. This qualitative behavior persists if $g(x)$ is nearly linear. If $f(t)$ is small, the periodic solution which is the perturbation of the identically zero solution of (1.1) when $f(t) \equiv 0$ is in $\mathscr{E}$.

However, for equations with large nonlinearities and large forcing there have been observed experimentally periodic solutions which are not odd-harmonic, i.e. they contain even harmonics. For examples of such experimental results, see the book by Hayashi [*2*]. For a mathematical treatment of a particular case, see the paper by Turrittin and Culmer [*5*].

Recently the author [*3*] in a study of piecewise linear equations of the form (1.1) found such solutions occurring in a natural way as the result of branching of an $\mathscr{E}$-solution. In the piecewise linear case it is possible to tell exactly for what values of the parameters of the equation such branching will occur. It turns out that for more general $g(x)$ it is possible in some cases to obtain similar results. It is the purpose of this paper to obtain the existence

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under certain restrictions of periodic solutions of equations of the form (1.1) which are not in $\mathscr{E}$.

The periodic solutions found are of two different types. One type is still an even function, but its Fourier series contains both even and odd harmonics It has a form similar to $\cos t + A \cos 2t$ with small A. The second type is an odd function about $t = \pi/2$ with Fourier series containing odd harmonics which are cosines and even harmonics which are sines. Its form is similar to $\cos t + A \sin 2t$ with small A.

It is not possible to say very much about the stability of the periodic solutions found. Assuming a small amout of damping, a periodic solution is either asymptotically stable or has a "saddle" type instability. The periodic solutions not in $\mathscr{E}$ are found to occur in pairs (if $x_1(t)$ is such a solution, so is $-x_1(t + \pi)$) and the two solutions of such a pair have the same stability, which is in turn opposite in character to the stability of a related $\mathscr{E}$-solution. It is necessary to make a special examination of each individual case to determine whether the solutions not in $\mathscr{E}$ are stable or unstable. In the specific cases known to the author they are stable (with small damping).

The underlying "cause" of the existence of such periodic solutions seems to be the presence of an even square integer in the range of $g'(x)$. An associated variation equation must have a nontrivial π-periodic solution, and this is impossible if the range of $g'(x)$ includes no even square integer. In the case studied in this paper, $g'(x)$ is confined between 1 and 9 and is made to pass through the value 4. The intimate association between square integers in the range of $g'(x)$ and branching phenomena is brought out in the study of piecewise linear systems [3].

It is possible to seek and predict periodic solutions not in $\mathscr{E}$ by the method of harmonic balance. Because of the results of Cesari [1] and Urable [6] this idea could also be used for existence proofs for such solutions. For equation (1.1) we would set $x = A \cos t + B \cos 2t + C \sin 2t$, substitute into (1.1), expand each side in Fourier series and equate coefficients of $\cos t$, $\cos 2t$, and $\sin 2t$. When the resulting equations for A, B, and C have solutions with B and C not both zero, a periodic solution not in $\mathscr{E}$ is predicted. It is sometimes necessary to assume a term $D \cos 3t$ in x in order to distinguish between the two kinds of such solutions.

2. A Family of $\mathscr{E}$-Solutions

We shall consider periodic solutions of the differential equation

$$x'' + g(x) = E \cos t, \tag{2.1}$$

where $g(x)$ is an odd function with $xg(x) > 0$ and $xg''(x) \geqslant 0$ for $x \neq 0$. We shall also assume that $g'''(x)$ is continuous. More important we assume that the range of variation of $g'(x)$ is limited as follows. Clearly $g'(x)$ is an even function which is nondecreasing for nonnegative x. We require that $1 < g'(0) < 4$, and that $4 < g'(\infty) < 9$.

With the hypotheses on $g(x)$ just listed it is possible to show that for each value of E there exists a unique $\mathscr{E}$-solution of (2.1). The proof of this is given in a paper by the author [4], and will not be given in detail here. The idea of the proof is to note that any $\mathscr{E}$-solution has $x'(0) = 0$ and $x(\pi/2) = 0$. It can be shown that if $x(t, A, E)$ is that solution of (2.1) with $x(0) = A$, $x'(0) = 0$, then $x(\pi/2, A, E)$ is for each E a strictly decreasing function of A with the derivative with respect to A bounded and also bounded away from zero. Hence for each E there is exactly one value of A for which $x(\pi/2, A, E) = 0$, and foı this A the solution $x(t, A, E)$ is an $\mathscr{E}$-solution.

Let $x(t, E)$ denote the unique $\mathscr{E}$-solution of (2.1). We shall wish to show that for at least one positive value of E the variation equation

$$y'' + g'(x(t, E))\, y = 0 \tag{2.2}$$

has an even π-periodic nontrivial solution, and that for at least one positive value of E, (2.2) has an odd π-periodic nontrivial solution.

LEMMA 1. *Let $x(t, E)$ be the unique $\mathscr{E}$-solution of (2.1). Then*

$$\lim_{|E|\to\infty} \left| \frac{x(t, E)}{E} - \frac{\cos t}{k^2 - 1} \right| = 0$$

uniformly in t, where $k^2 = g'(\infty)$.

Proof. Let $h(x) = k^2 x - g(x)$. Then for $x > 0$, $h(x) \geqslant 0$, $h'(x) = k^2 - g'(x) \geqslant 0$, $h''(x) = -g''(x) \leqslant 0$. Also both $h'(x)$ and $h(x)/x$ are nonincreasing and have the limit zero as $x \to \infty$.

Let $G(t, s)$ be the Green's function for the problem

$$z'' + k^2 z = f(t), \qquad z'(0) = 0, \qquad z(\pi/2) = 0.$$

$G(t, s)$ is readily computed; it is continuous and bounded, and we easily find that

$$| G(t, s)| \leqslant \frac{1}{k \mid \cos k\pi/2 \mid} = G_0 \text{ say.}$$

Equation (2.1) can be written in the form

$$x'' + k^2 x = E \cos t + h(x),$$

so that $x(t, E)$ is the unique solution of the integral equation

$$x(t) = \frac{E \cos t}{k^2 - 1} + \int_0^{\pi/2} G(t, s)\, h(x(s))\, ds. \tag{2.3}$$

Because of the periodicity properties of $x(t, E)$ we need only consider the interval $0 \leqslant t \leqslant \pi/2$. Let $M = \max |x(t)|$ on $0 \leqslant t \leqslant \pi/2$. It then follows from (2.3) that

$$M \geqslant |x(0)| \geqslant \frac{|E|}{k^2 - 1} - \frac{\pi}{2} G_0 h(M)$$

whence

$$\frac{|E|}{k^2 - 1} \leqslant \left(1 + \frac{\pi}{2} G_0 \frac{h(M)}{M}\right) M \leqslant \mu M,$$

where

$$\mu = \max \left(1 + \frac{\pi}{2} G_0 \frac{h(u)}{u}\right).$$

Thus $M \to \infty$ as $|E| \to \infty$. It follows from (2.3) that

$$M \leqslant \frac{|E|}{k^2 - 1} + \frac{\pi}{2} G_0 h(M),$$

so that

$$\frac{M}{|E|} \left[1 - \frac{\pi}{2} G_0 \frac{h(M)}{M}\right] \leqslant \frac{1}{k^2 - 1}.$$

Let M_0 be such that for, $M > M_0$, $(\pi/2) G_0\, h(M)/M < 1/2$, and let E_0 be so large that if $E > E_0$, $M > M_0$. Then it follows that for $E > E_0$,

$$\frac{M}{|E|} \leqslant \frac{2}{k^2 - 1}.$$

Finally (2.3) gives

$$\left| \frac{x(t, E)}{E} - \frac{\cos t}{k^2 - 1} \right| \leqslant \frac{\pi}{2} G_0 \frac{h(M)}{|E|} = \frac{\pi}{2} G_0 \frac{h(M)}{M} \frac{M}{|E|}.$$

Since $M/|E|$ is bounded for large E, and since as $E \to \infty$, $M \to \infty$ and $h(M)/M \to 0$, the conclusion of Lemma 1 follows. This completes the proof.

In further preparation for the study of the variation equation we prove

LEMMA 2. *For any number ω, $2 < \omega < k$, there exists a number E_1 such that if $|E| > E_1$,*

$$g'(x(t, E)) > \omega^2 \qquad 0 \leqslant t \leqslant \frac{\pi}{\omega}.$$

Proof. Let A_0 be such that if $|x| \geqslant A_0$, $g'(x) > \omega^2$. Let E_2 be such that if $|E| > E_2$,

$$\left| \frac{x(t)}{E} - \frac{\cos t}{k^2 - 1} \right| < \frac{1}{2} \frac{\cos \pi/\omega}{k^2 - 1} .$$

Then if $|E| > E_2$,

$$\frac{x(t)}{E} > \frac{\cos t}{k^2 - 1} - \frac{1}{2} \frac{\cos \pi/\omega}{k^2 - 1} ,$$

so that for $0 \leqslant t \leqslant \pi/\omega$,

$$\frac{x(t)}{E} \geqslant \frac{\cos \pi/\omega}{2(k^2 - 1)} .$$

Now let

$$E_1 = \max \left\{ E_2 , \frac{2(k^2 - 1) A_0}{\cos \pi/\omega} \right\} .$$

Then, since $E_1 \geqslant E_2$, when $|E| > E_1$,

$$|x(t)| \geqslant |E| \frac{\cos \pi/\omega}{2(k^2 - 1)} \qquad \left(0 \leqslant t \leqslant \frac{\pi}{\omega}\right) .$$

But, when

$$|E| > E_1 , |E| > \frac{2(k^2 - 1) A_0}{\cos \pi/\omega} ,$$

so that

$$|x(t)| > A_0 \qquad \left(0 \leqslant t \leqslant \frac{\pi}{\omega}\right) .$$

This shows that if $|E| > E_1$,

$$g'(x(t, E)) > \omega^2 \qquad \left(0 \leqslant t \leqslant \frac{\pi}{\omega}\right) .$$

This completes the proof of Lemma 2.

THEOREM 1. *Let $\phi(t)$ and $\psi(t)$ be the solutions of the variation equation*

$$y'' + g'(x(t, E))y = 0 \tag{2.2}$$

satisfying the initial conditions $\phi(0) = \psi'(0) = 1$, $\phi'(0) = \psi(0) = 0$. Then for at least one positive value of E (with $E \leqslant E_1$) $\phi'(\pi/2) = 0$, and $\phi(t)$ is an even, π-periodic, nontrivial solution of (2.2). Also for at least one positive value of E (with $E \leqslant E_1$) $\psi(\pi/2) = 0$, and $\psi(t)$ is an odd, π-periodic, nontrivial solution of (2.2).

Proof. For $E = 0$, $x(t, 0) \equiv 0$, and the variation equation is

$$y'' + g'(0)y = 0.$$

Since $1 < g'(0) < 4$, $\phi'(\pi/2) = -[g'(0)]^{1/2} \sin [g'(0)]^{1/2} \pi/2 < 0$, and $\psi(\pi/2) = [g'(0)]^{-1/2} \sin [g'(0)]^{1/2} \pi/2 > 0$.

For $E > E_1$, $g'(x(t, E)) > \omega^2$ for $0 \leqslant t \leqslant \pi/\omega$, and on this time interval, $\phi(t)$ and $\psi(t)$ oscillate more rapidly than $\cos \omega t$ and $\sin \omega t$. Hence at $t = \pi/\omega$, at which point $\cos \omega t$ has its first minimum, $\phi(t)$ has already passed its first minimum, and $\phi'(\pi/\omega) > 0$. Hence since $g'(x) > 0$, $\phi'(\pi/2) > 0$ as well. ($\phi(t)$ cannot have reached its second maximum by $\pi/2$ since $g'(x) < 16$). Similarly, $\psi(t)$ will have passed its first zero by $t = \pi/\omega$, but will not have reached its second zero before $t = \pi/2$, so that $\psi(\pi/2) < 0$.

Since $\phi'(\pi/2)$ has opposite signs for $E = 0$ and for $E > E_1$, there must be a value of $E(0 < E \leqslant E_1)$ such that $\phi'(\pi/2) = 0$. Similarly, there must be a value of E in this same interval for which $\psi(\pi/2) = 0$.

Since $g'(x(t, E))$ is even and π-periodic, $\phi'(\pi/2) = 0$ implies that $\phi(t)$ is even and π-periodic. Also $\psi(\pi/2) = 0$ implies that $\psi(t)$ is odd and π-periodic. This completes the proof of Theorem 1.

Remark. It is usually the case that the value of E for which $\phi'(\pi/2) = 0$ is not the same as the value of E for which $\psi(\pi/2) = 0$, although there are cases where they do coincide. Also, of course, there is no way of knowing that there is only one value of E with $\phi'(\pi/2) = 0$ and only one value of E with $\psi(\pi/2) = 0$. In the case of a piecewiselinear system [3] it is possible to know that there is just one value of E for each possibility, and it is also possible to tell which comes first. For the purposes of the remaining sections of this paper, we shall definitely assume that $\phi'(\pi/2) = 0$ and $\psi(\pi/2) = 0$ occur for different values of E.

3. Branching of the Solution $x(t, E)$

The periodic solutions which are not in $\mathscr{E}$, whose existence we shall prove, occur for values of E very near to values of E for which the variation equation

$$y'' + g'(x(t, E))y = 0$$

has a nontrivial π-periodic solution. If E^* is such a value of E, it is found under appropriate restrictions, which will be developed, that either for E immediately greater than E^* or for E immediately less than E^* (but not both) there will exist a pair of 2π-periodic solutions which are not in $\mathscr{E}$. These will both be even functions if the π-periodic solution of the variation equation is even, and will both be odd functions about $t = \pi/2$ if the π-periodic solution of the variation equation is odd.

The conditions to be imposed are first that for $E = E^*$ only one of the solutions $\phi(t)$ and $\psi(t)$ is periodic, and second that certain integrals not vanish. These requirements are necessary to ensure a reasonably simple structure for the branching. Our further work will depend on the discussion of two equations $F(\xi, \eta, E) = 0$ and $G(\xi, \eta, E) = 0$ near a point where their Jacobian vanishes. We can guarantee a simple structure if we require that important coefficients in the Taylor expansions of F and G do not vanish. Without looking ahead in this way we can see from the linear example $g(x) = 4x$ that the first condition is desirable. Here the variation equation is $y'' + 4y = 0$ for all values of E, and for every value of E both $\phi(t)$ and $\psi(t)$ are π-periodic. What is more, all solutions of $x'' + g(x) = E \cos t$ are 2π-periodic and, except for one special solution, no solution lies in $\mathscr{E}$.

The integrals that are required not to vanish involve certain partial derivatives of the solutions of $x'' + g(x) = E \cos t$ with respect to initial conditions or the parameter E, and it will be necessary to investigate these first. We shall treat in detail the case that the periodic solution of the variation equation for $E = E^*$ is even. The details when the periodic solution is odd are similar and will be summarized in Theorem 3.

We begin by examining the solutions of the variation equation

$$y'' + g'(x(t, E^*))y = 0, \tag{3.1}$$

where it is assumed that the solution $\phi(t)$, with initial conditions $\phi(0) = 1$, $\phi'(0) = 0$ satisfies $\phi'(\pi/2) = 0$ and so is even and π-peirodic. It is also assumed that the solution $\psi(t)$ with initial conditions $\psi(0) = 0$, $\psi'(0) = 1$ has $\psi(\pi/2) \neq 0$, so that it is odd, but not π-periodic.

If $\psi(t)$ is not periodic, it will have the form

$$\psi(t) = Kt\phi(t) + r(t) \tag{3.2}$$

where $K \neq 0$ and $r(t)$ is odd and π-periodic. If we set $\phi(\pi/2) = \alpha$ and $\psi(\pi/2) = \beta$, we know that $\alpha \neq 0$ since ϕ is nontrivial and $\phi'(\pi/2) = 0$, and our fundamental assumption is that $\beta \neq 0$. We observe in passing that because $1 < g'(x) < 9$, $\alpha < 0$. Because $r(t)$ is odd and π-periodic, $r(\pi/2) = 0$, so we can evaluate K in terms of α and β by setting $t = \pi/2$ in (3.2). The result is $K = (2\beta)/(\pi\alpha)$. It also proves convenient to introduce a third solution of equation (3.1), namely

$$q(t) = \psi(t) - \frac{K\pi}{2}\phi(t) = \psi(t) - \frac{\beta}{\alpha}\phi(t). \tag{3.3}$$

The important properties of $q(t)$ are that $q(\pi/2) = 0$ and that $q(\pi/2 + t) = -q(\pi/2 - t)$ which follow easily from (3.2) and the fact that $\phi(t)$ is even and

π-periodic. The behavior of $\phi(t)$ and $\psi(t)$ on $0 \leqslant t \leqslant 2\pi$ is summarized in the following table

t	$\phi(t)$	$\phi'(t)$	$\psi(t)$	$\psi'(t)$
0	1	0	0	1
$\pi/2$	α	0	β	$1/\alpha$
π	1	0	$2\beta/\alpha$	1
$3\pi/2$	α	0	3β	$1/\alpha$
2π	1	0	$4\beta/\alpha$	1

In the construction of the table the relation $\phi(t)\psi'(t) - \phi'(t)\psi(t) \equiv 1$ was used.

Let $x(t, \xi, \eta, E)$ be that solution of

$$x'' + g(x) = E \cos t \tag{3.4}$$

with the initial conditions $x(0) = \xi$, $x'(0) = \eta$. If $A^* = x(0, E^*)$ is the initial value of the periodic solution $x(t, E^*)$, then $x(t, E^*) \equiv x(t, A^*, 0, E^*)$. We define two functions, $F(\xi, \eta, E)$ and $G(\xi, \eta, E)$ by the relations

$$F(\xi, \eta, E) = x(2\pi, \xi, \eta, E) - \xi; \qquad G(\xi, \eta, E) = x'(2\pi, \xi, \eta, E) - \eta. \tag{3.5}$$

The solution $x(t, \xi, \eta, E)$ is 2π-periodic if and only if $F(\xi, \eta, E) = G(\xi, \eta, E) = 0$. In particular we know that $F(A^*, 0, E^*) = G(A^*, 0, E^*) = 0$.

We shall study the nature of the locus $F = G = 0$ in the neighborhood of the point $(A^*, 0, E^*)$. Part of the locus is the curve

$$\xi = x(0, E), \qquad \eta = 0, \qquad E = E \qquad -\infty < E < \infty, \tag{3.6}$$

which is the locus of initial conditions for the $\mathscr{E}$-solutions of (3.4) which we know to exist for every E. It will turn out that under appropriate conditions, the locus $F = G = 0$ will have a branching at the point $(A^*, 0, E^*)$. Points on the second branch will correspond to initial conditions for periodic solutions which are not in $\mathscr{E}$.

To investigate the branching of the locus $F = G = 0$ at the point $(A^*, 0, E^*)$ we shall need to evaluate several partial derivatives of the functions F and G at this point. This will require the computation of certain partial derivatives of the solution $x(t, \xi, \eta, E)$ with respect to the parameters ξ, η, and E evaluated for $\xi = A^*$, $\eta = 0$, $E = E^*$. Since all partial derivatives are evaluated for these particular parameter values, we do not mention this fact explicitly in what follows.

The partial derivatives of F and G will be found by evaluating corresponding partial derivatives of $x(t, \xi, \eta, E)$ for $t = 2\pi$. However, there are two partial derivatives of x which we need to study in more detail. These are the partial derivatives $x_E(t, A^*, 0, E^*)$ and $x_{\xi\xi}(t, A^*, 0, E^*)$.

$x_E(t, A^*, 0, E^*)$, which we write henceforth as $x_E(t)$, is the solution of the problem

$$y'' + g'(x(t, E^*))y = \cos t \qquad y(0) = y'(0) = 0, \tag{3.7}$$

and so is given by the formula

$$x_E(t) = \int_0^t [\phi(s)\,\psi(t) - \psi(s)\,\phi(t)] \cos s\, ds. \tag{3.8}$$

Since $x_E'(0) = 0$, the symmetry of the differential equation in (3.7) implies that $x_E(t)$ is an even function of t, and that $x_E'(t)$ is an odd function. If we compute $x_E'(\pi)$, we find

$$x_E'(\pi) = \int_0^\pi [\phi(s)\,\psi'(\pi) - \psi(s)\,\phi'(\pi)] \cos s\, ds = \int_0^\pi \phi(s) \cos s\, ds = 0,$$

since ϕ is π-periodic and $\cos s$ is in $\mathscr{E}$. $x_E(\pi) = x_E(-\pi)$ since x_E is even, and since $x_E'(-\pi) = -x_E'(\pi) = 0 = x_E'(\pi)$, the periodicity of the differential equation (3.7) shows that $x_E(t)$ is a 2π-periodic function. Formula (3.8) gives that

$$x_E(\pi/2) = x_E(3\pi/2) = \int_0^{\pi/2} [\beta\phi(s) - \alpha\psi(s)] \cos s\, ds = -\alpha \int_0^{\pi/2} q(s) \cos s\, ds,$$

$$x_E'(\pi/2) = -x_E'(3\pi/2) = 1/\alpha \int_0^{\pi/2} \phi(s) \cos s\, ds.$$

$$\begin{aligned} x_E(\pi) &= \int_0^\pi [(2\beta/\alpha)\,\phi(s) - \psi(s)] \cos s\, ds = -\int_0^\pi [q(s) - (\beta/\alpha)\,\phi(s)] \cos s\, ds \\ &= -\int_0^\pi q(s) \cos s\, ds = -2\int_0^{\pi/2} q(s) \cos s\, ds, \end{aligned}$$

since $\int_0^\pi \phi(s) \cos s\, ds = 0$ and $q(s) \cos s$ is even about $s = \pi/2$. If we define the constants B_1 and B_2 by the relations

$$B_1 = \int_0^{\pi/2} \phi(s) \cos s\, ds, \qquad B_2 = \int_0^{\pi/2} q(s) \cos s\, ds, \tag{3.9}$$

we have $x_E(\pi/2) = -\alpha B_2$, $x_E'(\pi/2) = B_1/\alpha$, $x_E(\pi) = -2B_2$.

The function $x_E(t)$ is even and 2π-periodic. It will be convenient to write it as the sum of an even π-periodic function and a function in $\mathscr{E}$. If we set

$$z(t) = \tfrac{1}{2}[x_E(t) + x_E(t + \pi)] \qquad h(t) = \tfrac{1}{2}[x_E(t) - x_E(t + \pi)]$$

then $z(t)$ and $h(t)$ are respectively π-periodic and in $\mathscr{E}$, and $x_E(t) = z(t) + h(t)$. Because of the symmetry properties of the differential equation (3.7), $z(t)$ is found to satisfy

$$z'' + g'(x(t, E^*))z = 0,$$

which implies that $z(t)$ is a constant multiple of $\phi(t)$. Setting $t = \pi$ we find that $z(t) \equiv -B_2\phi(t)$, so we finally have

$$x_E(t) = -B_2\phi(t) + h(t), \tag{3.10}$$

and the values can be summarized in the following table:

t	$x_E(t)$	$x'_E(t)$	$h(t)$	$h'(t)$
0	0	0	B_2	0
$\pi/2$	$-\alpha B_2$	$(1/\alpha)B_1$	0	$(1/\alpha)B_1$
π	$-2B_2$	0	$-B_2$	0
$3\pi/2$	$-\alpha B_2$	$-(1/\alpha)B_1$	0	$-(1/\alpha)B_1$
2π	0	0	B_2	0

The function $x_{\xi\xi}(t, A^*, 0, E^*)$ is the solution of the problem

$$y'' + g'(x(t, E^*))y = -g''(x(t, E^*))\phi(t)^2 \qquad y(0) = y'(0) = 0. \tag{3.11}$$

The right member, $-g''(x(t, E^*))\,\phi(t)^2$ is, like $\cos t$, a function in $\mathscr{E}$. Therefore, the analysis of (3.11) proceeds exactly as the analysis of (3.7). We define B_3 and B_4 as

$$\begin{aligned} B_3 &= \int_0^{\pi/2} g''(x(s, E^*))\,\phi(s)^3\,ds, \\ B_4 &= \int_0^{\pi/2} g''(x(s, E^*))\,\phi(s)^2\,q(s)\,ds. \end{aligned} \tag{3.12}$$

We find that $x_{\xi\xi}(t, A^*, 0, E^*)$ is even and 2π-periodic. It can be written as the sum of an even π-periodic function and a function in $\mathscr{E}$

$$x_{\xi\xi}(t, A^*, 0, E^*) = B_4\phi(t) + k(t) \tag{3.13}$$

and we have the following table of values:

t	$x_{\xi\xi}(t)$	$x'_{\xi\xi}(t)$	$k(t)$	$k'(t)$
0	0	0	$-B_4$	0
$\pi/2$	αB_4	$-(1/\alpha)B_3$	0	$-(1/\alpha)B_3$
π	$2B_4$	0	B_4	0
$3\pi/2$	αB_4	$(1/\alpha)B_3$	0	$(1/\alpha)B_3$
2π	0	0	$-B_4$	0

We now compute the needed partial derivatives of $F(\xi, \eta, E)$ and $G(\xi, \eta, E)$ at the point $(A^*, 0, E^*)$. We need to define three more integrals:

$$\begin{aligned} B_5 &= \int_0^{\pi/2} g''(x(s, E^*))\, \phi(s)^2\, h(s)\, ds \\ B_6 &= \int_0^{\pi/2} g''(x(s, E^*))\, \phi(s)^2\, k(s)\, ds \\ B_7 &= \int_0^{\pi/2} g'''(x(s, E^*))\, \phi(s)^4\, ds. \end{aligned} \tag{3.14}$$

LEMMA 3. *At the point* $(A^*, 0, E^*)$ *we have the following:*

$$\begin{gathered} F_\xi = 0, \qquad F_\eta = 4\beta/\alpha, \qquad F_E = 0, \qquad F_{\xi\xi} = 0, \\ G_\xi = 0, \qquad G_\eta = 0, \qquad G_E = 0, \qquad G_{\xi\xi} = 0 \\ G_{\xi\eta} = 0, \qquad G_{\xi E} = -4B_5\,, \qquad G_{EE} = 8B_2B_5 \qquad F_{\xi E} + G_{\eta E} = -(16\beta/\alpha)\, B_5 \\ G_{\xi\xi\xi} = -12B_6 - 4B_7\,. \end{gathered}$$

Proof. The partial derivatives in the first two lines of Lemma 3 follow directly from the preceding tables:

$$\begin{aligned} F_\xi &= x_\xi(2\pi) - 1 &&= \phi(2\pi) - 1 = 0, \\ F_\eta &= x_\eta(2\pi) &&= \psi(2\pi) = 4\beta/\alpha, \\ F_E &= x_E(2\pi) &&= 0, \\ F_{\xi\xi} &= x_{\xi\xi}(2\pi) &&= 0, \\ G_\xi &= x'_\xi(2\pi) &&= \phi'(2\pi) = 0, \\ G_\eta &= x'_\eta(2\pi) - 1 &&= \psi'(2\pi) - 1 = 0, \\ G_E &= x'_E(2\pi) &&= 0, \\ G_{\xi\xi} &= x'_{\xi\xi}(2\pi) &&= 0. \end{aligned}$$

The computation of the remaining partial derivatives is not difficult but is somewhat lengthy. To illustrate a computation, we compute $G_{\xi E}$.

$x_{\xi E}(t, A^*, 0, E^*)$ is the solution of

$$y'' + g'(x(t, E^*))y = -g''(x(t, E^*))\, \phi(t)\, x_E(t), \qquad y(0) = y'(0) = 0.$$

It is given by

$$x_{\xi E}(t) = -\int_0^t [\phi(s)\, \psi(t) - \psi(s)\, \phi(t)]\, g''(x(s, E^*))\, \phi(s)\, x_E(s)\, ds.$$

Hence

$$G_{\xi E} = x'_{\xi E}(2\pi) = -\int_0^{2\pi} g''(x(s, E^*))\, \phi(s)^2\, x_E(s)\, ds$$

because $\phi'(2\pi) = 0$, $\psi'(2\pi) = 1$. Finally, since $x_E(s) = -B_2\phi(s) + h(s)$, and because the periodic function $g''(x(s, E^*))\, \phi(s)^3$ has mean value zero,

$$G_{\xi E} = -\int_0^{2\pi} g''(x(s, E^*))\, \phi(s)^2\, h(s)\, ds = -4B_5\,.$$

The final integrand is even and π-periodic, so its integral from 0 to 2π is four times its integral from 0 to $\pi/2$. The remaining partial derivatives of Lemma 3 are evaluated similarly. This completes the proof of Lemma 3.

We are now in a position to study the locus $F = G = 0$ in the neighborhood of $(A^*, 0, E^*)$. Because $F_\eta(A^*, 0, E^*) \neq 0$ ($\beta \neq 0$ by hypothesis), we may solve the equation $F(\xi, \eta, E) = 0$ for η as a function of ξ and E in the neighborhood of $\xi = A^*$, $E = E^*$. If the result is written $\eta = H(\xi, E)$, the results of Lemma 3 give at (A^*, E^*):

$$H = H_\xi = H_E = 0. \tag{3.15}$$

If we then define the function $J(\xi, E)$ in the neighborhood of (A^*, E^*) by the relation

$$J(\xi, E) = G(\xi, H(\xi, E), E),$$

the locus $J(\xi, E) = 0$ is the projection onto a plane $\eta =$ const. of the locus $F = G = 0$, and the projection is one to one in the neighborhood of $(A^*, 0, E^*)$. At (A^*, E^*) the derivatives of J are found from Lemma 3 to be:

$$\begin{aligned} &J = J_\xi = J_E = J_{\xi\xi} = 0, \qquad J_{\xi E} = -4B_5\,, \\ &J_{EE} = 8B_2B_5\,, \qquad J_{\xi\xi\xi} = -12B_6 - 4B_7\,. \end{aligned} \tag{3.16}$$

Now the principal terms of $J(\xi, E)$ near (A^*, E^*) are:

$$\tfrac{1}{2}J_{EE}(E - E^*)^2 + J_{\xi E}(E - E^*)(\xi - A^*) + \tfrac{1}{6}J_{\xi\xi\xi}(\xi - A^*)^3 =$$
$$4B_2B_5(E - E^*)^2 - 4B_5(E - E^*)(\xi - A^*) - (2B_6 + \tfrac{2}{3}B_7)(\xi - A^*)^3.$$

These terms show that in the neighborhood of (A^*, E^*), assuming that all three of the coefficients are nonzero, the locus $J = 0$ consists of two branches, one tangent to the line $\xi - A^* = B_2(E - E^*)$, with the other tangent to the line $E = E^*$ and approximated by the parabola $E - E^* = -((3B_6 + B_7)/(6B_5))(\xi - A^*)^2$. Hence the locus $F = G = 0$ also has two branches through the point $(A^*, 0, E^*)$. The tangents to both these branches lie in the plane $\eta = 0$ because $F_\xi = F_E = 0$. The tangents are the lines $\xi - A^* = B_2(E - E^*)$, $\eta = 0$ and $E = E^*$, $\eta = 0$.

We can identify the first of these branches, which exists for values of E on both sides of E^*, as the locus (3.6) of initial conditions for $\mathscr{E}$-solutions. This branch lies wholly in the plane $\eta = 0$.

The second branch, which exists only for $E \geqslant E^*$ or for $E \leqslant E^*$ according as $(3B_6 + B_7)/(6B_5)$ is negative or positive, is definitely distinct from the first branch and corresponds to initial conditions of 2π-periodic solutions which are not in $\mathscr{E}$. (We know that for each value of E there is exactly one $\mathscr{E}$-solution.)

THEOREM 2. *In the differential equation*

$$x'' + g(x) = E \cos t \tag{3.4}$$

let $g(x)$ be an odd function with $xg(x)$ and $xg''(x)$ positive for $x \neq 0$. Let $g'''(x)$ be continuous. Let $1 < g'(0) < 4$, $4 < g'(\infty) < 9$. Let E^* be such that the variation equation

$$y'' + g'(x(t, E^*))y = 0 \tag{3.1}$$

has a nontrivial, even, π-periodic solution but does not have a nontrivial, odd, π-periodic solution, where $x(t, E)$ is the unique $\mathscr{E}$-solution of (3.4). *Let the integrals B_2 and B_5 and the combination $3B_6 + B_7$ be different from zero.*

Then there exist near $E = E^*$ two even 2π-periodic solutions of (3.4) *which are not in $\mathscr{E}$, $x_1(t)$ and $x_2(t) \equiv -x_1(t + \pi)$. These exist for $E \geqslant E^*$ only or for $E \leqslant E^*$ only according as $(3B_6 + B_7)/(6B_5)$ is negative or positive.*

Proof. The existence of the value E^* follows from Theorem 1. The additional hypotheses guarantee the validity of the preceding geometric analysis and thus the existence for E immediately less or immediately greater than E^* of the second branch of the locus $F = G = 0$, and thus the existence of additional 2π-periodic solutions of (3.4) for such values of E. As was remarked earlier the solutions corresponding to points on the second branch of $F = G = 0$ cannot be in $\mathscr{E}$ because only points on the first branch correspond to such solutions.

It remains to prove that the solutions corresponding to points on the second branch are even functions. Let $x_1(t)$ be a solution corresponding to a point on the second branch. By the symmetries of the differential equation (3.4), $x_2(t) \equiv -x_1(t + \pi)$, $x_3(t) \equiv x_1(-t)$, and $x_4(t) \equiv -x_1(\pi - t)$ are also 2π-periodic solutions. All four are not identical since $x_1(t)$ is not in $\mathscr{E}$. All four are not distinct since there are exactly two solutions for E near E^* on the second branch. Therefore, they must be equal in pairs. Since at $t = 0$,

$x_1 = x_3$ and $x_2 = x_4$, it must be that $x_1(t) \equiv x_3(t)$ and $x_2(t) \equiv x_4(t)$. These identities show that $x_1(t)$ and $x_2(t)$ are even functions of t. This completes the proof.

Remarks. 1. The same reasoning would hold if the right member of (3.4) were of the form $Ef(t)$ with $f(t)$ in $\mathscr{E}$, so that the loss in generality occasioned by the use of $\cos t$ is very slight.

2. The requirements that B_2, B_5, and the combination $3B_6 + B_7$ not vanish are usually fulfilled. In cases that one or more of these expressions vanish, a deeper analysis will often give similar results.

3. The result could also be proved if the range of $g'(x)$ is described by

$$(2n-1)^2 < g'(0) < 4n^2, \qquad 4n^2 < g'(\infty) < (2n+1)^2$$

for any integer $n \geqslant 1$. The reasoning is exactly the same. Also the result can be obtained if $g(x)$ is of "softening" character with $g'(x)$ decreasing steadily in an interval contained between two odd square integers and including the intermediate even square integer. The proof of Lemma 2 has to be modified in this case. On the other hand it seems necessary to exclude odd square integers from the range of $g'(x)$ and also to insist that $g'(x)$ be monotonic for $x > 0$.

A similar theorem can be developed for the case that the variation equation (3.4) has an odd-π-periodic solution. For this it is most convenient to shift the time origin to $\pi/2$. The differential equation then becomes

$$x'' + g(x) = E \sin t \tag{3.17}$$

which has, for each value of E, a 2π-periodic solution which is both an odd function and an odd-harmonic function. Using a notation similar to that in earlier sections we speak of a solution in $\mathscr{O}$ or an $\mathscr{O}$-solution. This time if $x(t, \xi, \eta, E)$ is that solution of (3.17) with $x = \xi$, $x' = \eta$ at $t = 0$, the unique $\mathscr{O}$-solution has $x(0) = 0$, $x'(\pi/2) = 0$.

Let this solution as before be denoted by $x(t, E)$. If E^{**} is a value of E for which the variation equation has an odd, π-periodic solution, let $V^{**} = x'(0, E^{**})$. Then we have to study the locus $F = G = 0$ in the neighborhood of $(0, V^{**}, E^{**})$.

If $\phi(t)$ and $\psi(t)$ are the standard solutions of the variation equation, we have that $\psi(\pi/2) = 0$, and we assume that $\phi'(\pi/2) \neq 0$. The function $\tilde{q}(t)$ corresponding to $q(t)$ is defined by $\tilde{q}(t) = \phi(t) - (\tilde{\beta}/\tilde{\alpha})\,\psi(t)$, $\tilde{\beta} = \phi'(\pi/2)$, $\tilde{\alpha} = \psi'(\pi/2)$. $q(t)$ is then an even function about $\pi/2$. It is found this time that $x_E(t)$ and $x_{\eta\eta}(t)$ are odd, 2π-periodic functions, and they can be written as

$$x_E(t) = C_2\psi(t) + \tilde{h}(t),$$
$$x_{\eta\eta}(t) = -C_4\psi(t) + \tilde{k}(t),$$

where

$$C_1 = \int_0^{\pi/2} \psi(s) \sin s \, ds, \qquad C_2 = \int_0^{\pi/2} \tilde{q}(s) \sin s \, ds,$$

$$C_3 = \int_0^{\pi/2} g''(x(s, E^{**}))\, \psi(s)^3 \, ds, \qquad C_4 = \int_0^{\pi/2} g''(x(s, E^{**}))\, \psi(s)^2 \, \tilde{q}(s) \, ds,$$

$$C_5 = \int_0^{\pi/2} g''(x(s, E^{**}))\, \psi(s)^2 \, \hat{h}(s) \, ds, \qquad C_6 = \int_0^{\pi/2} g''(x(s, E^{**}))\, \psi(s)^2 \, \hat{k}(s) \, ds,$$

$$C_7 = \int_0^{\pi/2} g'''(x(s, E^{**}))\, \psi(s)^4 \, ds.$$

This time we solve $G(\xi, \eta, E) = 0$ for ξ as a function of η and E. Let the result be $\xi = H(\eta, E)$. Then let $J(\eta, E) = F(H(\eta, E), \eta, E)$. The principal terms of $J(\eta, E)$ are:

$$\tfrac{1}{2} J_{EE}(E - E^{**})^2 + J_{\eta E}(E - E^{**})(\eta - V^{**}) + \tfrac{1}{6} J_{\eta\eta\eta}(\eta - V^{**})^3 =$$
$$4C_2 C_5 (E - E^{**})^2 + 4C_5(E - E^{**})(\eta - V^{**}) + (2C_6 + \tfrac{2}{3} C_7)(\eta - V^{**})^3.$$

Again there is a branching of the locus $F = G = 0$ at $(0, V^{**}, E^{**})$. One branch corresponds to the $\mathcal{O}$-solutions, the other to solutions which are not in $\mathcal{O}$.

THEOREM 3. *In the equation*

$$x'' + g(x) = E \sin t, \tag{3.17}$$

let $g(x)$ satisfy the hypotheses listed in Theorem 2. Let the variation equation for $E = E^{**}$ have a nontrivial odd, π-periodic solution and no nontrivial even π-periodic solution. Let the integrals C_2, C_5, and the combination $3C_6 + C_7$ be different from zero.

Then there exist near $E = E^{**}$ two odd, 2π-periodic solutions of (3.17) *which are not in $\mathcal{O}$, $x_1(t)$ and $x_2(t) \equiv -x_1(t + \pi)$. These exist for $E \geqslant E^{**}$ only or for $E \leqslant E^{**}$ only according as $(3C_6 + C_7)/(6C_5)$ is negative or positive.*

Remark. The functions $x_1(t + \pi/2)$ and $x_2(t + \pi/2)$ are solutions of (3.4) which are odd functions about $t = \pi/2$.

4. REMARKS ON STABILITY

It does not seem possible to make definite statements about the stability of the periodic solutions not in $\mathscr{E}$ found in Section 3. However, there are some conditions under which their stability can be asserted.

Since the equation (3.4) has no damping term, the stability problem for its solutions is difficult. The characteristic multipliers of the variation equation relative to any periodic solution of (3.4) always have product $+1$. If they are real and unequal, the corresponding periodic solution is directly unstable. If they are real and equal, we are in the situation of a branching point. If the characteristic multipliers are nonreal with absolute value one, the solutions of the variation equation are stable, and nothing can be said about the stability of the periodic solution of the original equation. However, in this latter case, if the equation is then perturbed with a small positive damping, the periodic solution becomes asymptotically stable. When we speak of a periodic solution of (3.4) as being *stable*, it will always be meant that the characteristic multipliers of the variation equation are nonreal and that the solution is asymptotically stable in the presence of small positive damping. Note that when the characteristic multipliers of the variation equation are real and unequal, the direct instability persists even with small damping.

Let us first consider the stability of the $\mathscr{E}$-solutions $x(t, E)$. We shall show that, under our assumption that $B_5 \neq 0$, the character of the stability will change as E passes through E^*, that is, for E on one side of E^* the characteristic multipliers will be real and unequal; on the other side of E^* they will be nonreal with absolute value 1. If we use the notation of Section 3, the sum of the characteristic multipliers of the variation equation with respect to the solution $x(t, E)$ will be

$$x_\xi(2\pi, A, 0, E) + x'_\eta(2\pi, A, 0, E),$$

where $A = x(0, E)$. When $E = E^*$, this sum has the value 2. We can evaluate the derivative of this sum at $E = E^*$. Using the F and G notation, we find the derivative with respect to E of the sum of the characteristic multipliers to be

$$F_{\xi\xi} \frac{dA}{dE} + F_{\xi E} + G_{\xi\eta} \frac{dA}{dE} + G_{\eta E},$$

all derivatives being evaluated at $(A^*, 0, E^*)$. Since at this point $F_{\xi\xi} = G_{\xi\eta} = 0$, this reduces to

$$F_{\xi E} + F_{\eta E} = -\frac{16\beta}{\alpha} B_5 . \tag{4.1}$$

Now the characteristic multipliers are real and unequal if and only if their sum is greater than 2, and this gives direct instability. The characteristic multipliers are nonreal and of absolute value 1 if and only if their sum is less than 2, and this gives stability. The sign of the derivative (4.1) gives the behavior of this sum near $E = E^*$, so we can say that $x(t, E)$ is stable for

$E < E^*$ and directly unstable for $E > E^*$ if $\alpha\beta\, B_5 < 0$, and $x(t, E)$ is stable for $E > E^*$ and directly unstable for $E < E^*$ if $\alpha\beta\, B_5 > 0$.

The periodic solutions which are not in $\mathscr{E}$ exist only on one side of E^*, and where they exist, their stability will be opposite to that of $x(t, E)$. To see this we need only consider for each E near E^* the index of a small circle with center at $(A^*, 0)$ in the $\xi\eta$-plane. As E varies, this index remains constant, while the index of the point $(x(0, E), x'(0, E))$ changes sign as E passes through E^*. Thus if the periodic solutions not in $\mathscr{E}$ exist for values of E which make $x(t, E)$ directly unstable, the solutions not in $\mathscr{E}$ are stable, and vice-versa. By Theorem 2 the periodic solutions not in $\mathscr{E}$ exist for $E > E^*$ if $B_5(3B_6 + B_7) < 0$, and they exist for $E < E^*$ if $B_5(3B_6 + B_7) > 0$. Combining these results we obtain

THEOREM 4. *The periodic solutions not in $\mathscr{E}$ found in Theorem 2 are stable if and only if $\alpha\beta(3B_6 + B_7) > 0$.*

Remark. A corresponding result holds for the periodic solutions not in $\mathscr{O}$ found in Theorem 3. The condition for their stability is $\tilde{\alpha}\tilde{\beta}(3C_6 + C_7) > 0$.

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